Combinatorics

Quinten Jin, Emre Kocaman

Massachusetts Institute of Technology

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Example

40 people speak French and 30 people speak German in a classroom. 20 people speak both. Everyone speaks 1 language at least. How many people are there?
Answer: 50
When we add 40 and 30, we get 70. However, we count the students who speak both two times. Thus, we have to subtract the number of students who speak both. When we subtract 20 from 70, we get 50. We can state our problem as $|A| + |B| - |A \cap B| = |A \cup B|$ where $|A|$ represents French speakers and $|B|$ represents German speakers. It can be shown in Venn Diagram as follows:

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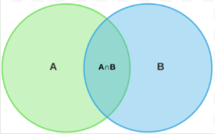

Theorem (Principle of Inclusion and Exclusion)

PIE stands for the Principle of Inclusion and Exclusion and is used to avoid over counting and over subtracting. The generalized formula for PIE is:

$$\left|\bigcup_{i=1}^{n} S_k\right| = \sum_{I\subseteq[n]} (-1)^{(|I|+1)} \left|\bigcap_{i\in I} S_i\right|$$

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The language problem we just gave is an example for PIE. In the formula, the left side is $| S_1 | \cup | S_2 | \cup | S_3 | \cdots | S_k |$. The right side indicates that if the number of intersecting sets is odd, we add the intersection. If the number of intersecting sets is even, we subtract the intersection. For example, the right side says that $| S_1 \cap S_2 |$ is subtracted and $| S_1 \cap S_2 \cap S_3 |$ is added.

Now, we are going to use PIE in a hard problem which seems irrelevant to PIE.

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5.The number of alternatives we don't want is $\mid S_1 \mid + \mid S_2 \mid + \mid S_3 \mid + \mid S_4 \mid$ $-(\mid S_1 \cap S_2 \mid + \mid S_1 \cap S_3 \mid \cdots) + (\mid S_1 \cap S_2 \cap S_3 \cdots) - (\mid S_1 \cap S_2 \cap S_3 \cap S_4)$.

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8. We get $2 \times 4 \times 6 \times 10$.

Theorem (Catalan Numbers)

Let's say there are n (+1)'s and n (-1)'s. When these are arranged and added from left to right, the sum is always zero or bigger. That means that the number of (-1)'s is never larger than the number of (+1)'s. C_n *is the number of arrangements which satisfy that. The closed formula for* C_n *is* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

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2.(+1),(-1),(+1),(-1) is okay.
3.(-1),(-1),(+1),(+1) is NOT okay.
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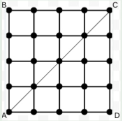
Here, open parenthesis is same with $+1$ and closed parenthesis is same with -1.
For example: $(+1)$,(-1),$(+1)$,(-1),$(+1)$,(-1) is same with ( ) ( ) ( ).
Thus, all we have to do is placing n with 3 in the closed formula. $\frac{1}{4}\binom{6}{3} = 5$.

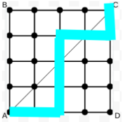
Example

In a 4×4 map, a car has to go from A to C. It can touch the diagonal but cannot pass it. It can only move to right and up. How many ways are there?



Answer: 336

In order not to pass the diagonal, the number of vertical moves shouldn't pass the number of horizontal moves. There should be 4 vertical and 4 horizontal moves. The vertical moves should be same as -1 and horizontal moves should be $+1$. For example, $(+1)$,$(+1)$,(-1),(-1),(-1),$(+1)$,$(+1)$,(-1) is equivalent to:



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- **Case 1: Horizontal Placement.** Placing a horizontal domino covers two consecutive cells in one row, requiring another horizontal domino directly beneath it. This reduces the problem to $F(n-2)$.

- **Case 2: Vertical Placement.** Placing a vertical domino covers one cell in each row, leaving a smaller $2 \times (n-1)$ board, hence, $F(n-1)$.

Recurrence Relation

The recursion for this problem is derived as:

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Consequently, $F(n)$ follows the Fibonacci sequence shifted by one position.

Challenge

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Hints: (1. Use the recursive formula $F_n = F_{n-1} + F_{n-2}$.
2. Find a geometric sequence which satisfies the recursive formula. (A geometric sequence is $a, ar, ar^2, ar^3 \cdots$) Then solve for $X^n = X^{n-1} + X^{n-2}$).

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1. We should find a geometric sequence which satisfies the recursive formula so we write $X^n = 4X^{n-1} - 3X^{n-2}$.
2. $(x-1)(x-3) = 0$. $x = 1$ and $x = 3$.
3. Since there are two possible geometric sequences, we algebraically combine them and set up the equations as $3 = A + B$ and $7 = A + 3B$. (3 is X_0 and 7 is X_1)
4.We find $A = 1$ and $B = 2$. Thus, the explicit formula of this recursive formula is $X_n = 1 + 2(3)^n$.

The method we used in our example is the same with the one used in proof of the Fibonacci explicit formula.

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Thus, the distribution of sums is: $[3, 4, 4, 5, 5, 5, 6, 6, 7]$.

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Simplifying this product gives:

$$G(x) = x^3 + 2x^4 + 3x^5 + 2x^6 + x^7$$